The Clifford-Fourier Integral Kernel in Even Dimensional Euclidean Space

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Abstract

Recently, we devised a promising new multi-dimensional integral transform within the Clifford analysis setting, the so-called Fourier-Bessel transform. In the specific case of dimension two, it coincides with the Clifford-Fourier transform introduced earlier as an operator exponential. Moreover, the L_2-basis elements, consisting of generalized Clifford-Hermite functions, appear to be simultaneous eigenfunctions of both integral transforms. In the even dimensional case, this allows us to express the Clifford-Fourier transform in terms of the Fourier-Bessel transform, leading to a closed form of the Clifford-Fourier integral kernel.

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1. Introduction

The *Fourier transform* is by far the most important integral transform. Since its introduction by Fourier in the early 1800s, it has remained an indispensable and stimulating mathematical concept that is at the core of the highly evolved branch of mathematics called *Fourier analysis*. It has found use in innumerable applications and has become a fundamental tool in engineering sciences, thanks to the generalizations extending the class of Fourier transformable functions and to the development of efficient algorithms for computing the discrete version of it.

The second player in this paper is *Clifford analysis*. It is a function theory for functions defined in Euclidean space $\mathbb{R}^m$ and taking values in the real Clifford algebra $\mathbb{R}_{0,m}$, constructed over $\mathbb{R}^m$. A Clifford algebra is an associative but non-commutative algebra with zero divisors, which combines the algebraic properties of the reals, the complex numbers and the quaternions with the geometric properties of a Grassmann algebra.
During the past 50 years, Clifford analysis has gradually developed into a comprehensive

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theory offering a direct, elegant and powerful generalization to higher dimension of the theory of holomorphic functions in the complex plane. In its most simple but still useful setting, flat m-dimensional Euclidean space, Clifford analysis focuses on monogenic functions, i.e. null solutions of the Clifford vector-valued Dirac operator $\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$, where $(e_1, \ldots, e_m)$ forms an orthogonal basis for the quadratic space $\mathbb{R}^{0,m}$ underlying the construction of the real Clifford algebra $\mathbb{R}_{0,m}$. Monogenic functions have a special relationship with harmonic functions of several variables in that they are refining their properties. The reason is that, as does the Cauchy-Riemann operator in the complex plane, the rotation-invariant Dirac operator factorizes the m-dimensional Laplace operator. At the same time, Clifford analysis offers the possibility of generalizing one-dimensional mathematical analysis to higher dimension in a rather natural way by encompassing all dimensions at once, in contrast to the traditional approach, where tensor products of one-dimensional phenomena are taken.

It is precisely this last qualification of Clifford analysis which has been exploited in [1] and [2] to construct a genuine multi-dimensional Fourier transform within the context of Clifford analysis. This so-called *Clifford-Fourier transform* is briefly discussed in Section 3. It is given in terms of an operator exponential or, alternatively, by a series representation. Particular attention is directed towards the two-dimensional case, since then the Clifford-Fourier kernel can be written in a closed form. Note that we have not succeeded yet in obtaining such a closed form in arbitrary dimension.
In [3] we introduced another promising multi-dimensional integral transform within the language of Clifford analysis, the so-called *Fourier-Bessel transform* (see Section 4). It appears that in the two-dimensional case, it coincides with the above mentioned Clifford-Fourier transform. Moreover, it satisfies operational formulae which are similar to those of the classical multi-dimensional Fourier transform. Furthermore, as is also the case for the Clifford-Fourier transform, the L_2-basis elements consisting of generalized Clifford-Hermite functions appear to be eigenfunctions of the Fourier-Bessel transform. The fact that the L_2-basis elements are simultaneous eigenfunctions of the Clifford-Fourier and the Fourier-Bessel transform will allow us, in the even dimensional case, to express the Clifford-Fourier transform in terms of the Fourier-Bessel transform (see Section 5), which leads to a closed form of the Clifford-Fourier integral kernel (see Section 6). To make the paper self-contained a section on definitions and basic properties of Clifford algebra and Clifford analysis is included (Section 2).

2. The Clifford Analysis Toolkit

Clifford analysis (see e.g. [4], [5], [6] and [7]) offers a function theory which is a higher dimensional analogue of the theory of the holomorphic functions of one complex variable.

The functions considered are defined in $\mathbb{R}^m$ $(m > 1)$ and take their values in the Clifford algebra $\mathbb{R}_{0,m}$ or its complexification $\mathbb{C}_m = \mathbb{R}_{0,m} \otimes \mathbb{C}$. If $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$, then a basis for the Clifford algebra $\mathbb{R}_{0,m}$ or $\mathbb{C}_m$ is given by all possible products of basis vectors $(e_A : A \subset \{1, \ldots, m\})$ where $e_\emptyset = 1$ is the identity element. The non-commutative multiplication in the Clifford algebra is governed by the rules: $e_j e_k + e_k e_j =$

$-2\delta_{j,k}$ $(j, k = 1, \dots, m)$.

Conjugation is defined as the anti-involution for which $\overline{e_j} = -e_j$ $(j = 1, \dots, m)$. In case of $\mathbb{C}_m$, the Hermitean conjugate of an element $\lambda = \sum_A \lambda_A e_A$ $(\lambda_A \in \mathbb{C})$ is defined by $\lambda^\dagger = \sum_A \lambda_A^c \ \overline{e_A}$, where λ_A^c denotes the complex conjugate of λ_A. This Hermitean conjugation leads to a Hermitean inner product and its associated norm on $\mathbb{C}_m$ given respectively by

$$(\lambda, \mu) = [\lambda^\dagger \mu]_0 \qquad \text{and} \qquad |\lambda|^2 = [\lambda^\dagger \lambda]_0 = \sum_A |\lambda_A|^2 \quad ,$$

where $[\lambda]_0$ denotes the scalar part of the Clifford element λ.

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebras $\mathbb{R}_{0,m}$ and $\mathbb{C}_m$ by identifying the point $(x_1, \dots, x_m)$ with the vector variable $\underline{x}$ given by $\underline{x} = \sum_{j=1}^m e_j x_j$. The product of two vectors splits up into a scalar part (the inner product up to a minus sign) and a so-called bivector part (the wedge product):

$$\underline{x}\,\underline{y} = \underline{x} \cdot \underline{y} \ + \underline{x} \wedge \underline{y} \quad ,$$

where

$$\underline{x} \cdot \underline{y} = - < \underline{x}, \underline{y} > = - \sum_{j=1}^m x_j y_j \qquad \text{and} \qquad \underline{x} \wedge \underline{y} = \sum_{i=1}^m \sum_{j=i+1}^m e_i e_j (x_i y_j - x_j y_i) \quad .$$

Note that the square of a vector variable $\underline{x}$ is scalar-valued and equals the norm squared up to a minus sign: $\underline{x}^2 = - < \underline{x}, \underline{x} > = -|\underline{x}|^2 = -r^2$.
Moreover, one can verify (see [8]) that for all $\underline{x}$, $\underline{t} \in \mathbb{R}^m$ the following formula holds:

$$(\underline{x} \wedge \underline{t})^2 = -|\underline{x} \wedge \underline{t}|^2 = (< \underline{x}, \underline{t} >)^2 - |\underline{x}|^2 |\underline{t}|^2 \quad . \tag{1}$$

The central notion in Clifford analysis is the notion of monogenicity, a notion which is the multi-dimensional counterpart to that of holomorphy in the complex plane. A function $F(x_1, \dots, x_m)$ defined and continuously differentiable in an open region of $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$ or $\mathbb{C}_m$, is called left monogenic in that region if $\partial_{\underline{x}}[F] = 0$. Here $\partial_{\underline{x}}$ is the Dirac operator in $\mathbb{R}^m$: $\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$, an elliptic, rotation-invariant, vector differential operator of the first order, which may be looked upon as the "square root" of the Laplace operator in $\mathbb{R}^m$: $\Delta_m = -\partial_{\underline{x}}^2$. This factorization of the Laplace operator establishes a special relationship between Clifford analysis and harmonic analysis in that monogenic functions refine the properties of harmonic functions.

In the sequel the monogenic homogeneous polynomials will play an important role. A left monogenic homogeneous polynomial P_k of degree k $(k \geq 0)$ in $\mathbb{R}^m$ is called a left solid inner spherical monogenic of order k. The set of all left solid inner spherical monogenics of order k will be denoted by $M_\ell^+(k)$. The dimension of $M_\ell^+(k)$ is given by

$$\dim\big(M_\ell^+(k)\big) = \binom{m+k-2}{m-2} = \frac{(m+k-2)!}{(m-2)!\ k!} \quad .$$

These left solid inner spherical monogenics are polynomial eigenfunctions of the so-called angular Dirac operator given by

$$\Gamma = -\underline{x} \wedge \partial_{\underline{x}} = -\sum_{i=1}^{m} \sum_{j=i+1}^{m} e_i e_j (x_i \partial_{x_j} - x_j \partial_{x_i}) \quad ,$$

i.e.

$$\Gamma[P_k] = -k P_k \quad , \quad P_k \in M_\ell^+(k) \quad . \tag{2}$$

This angular Dirac operator acts only on the angular co-ordinates. In Section 5 and 6 we will also use the following formula:

$$\Gamma[\underline{x} P_k] = (k + m - 1)\ \underline{x}\ P_k \quad , \quad P_k \in M_\ell^+(k) \quad . \tag{3}$$

The set

$$\phi_{s,k,j}(\underline{x}) = \frac{2^{m/4}}{(\gamma_{s,k})^{1/2}}\ H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right) \tag{4}$$

$s, k \in \mathbb{N},\ j \leq \dim\big(M_\ell^+(k)\big)$, constitutes an orthonormal basis for the space $L_2(\mathbb{R}^m)$ of square integrable functions. Here $\big\{P_k^{(j)}(\underline{x});\ j \leq \dim\big(M_\ell^+(k)\big)\big\}$ denotes an orthonormal basis of $M_\ell^+(k)$ and $\gamma_{s,k}$ a real constant depending on the parity of s. The polynomials $H_{s,k}(\underline{x})$ are the so-called generalized Clifford-Hermite polynomials introduced by Sommen in [9]; they are a multi-dimensional generalization to Clifford analysis of the classical Hermite polynomials on the real line. Note that $H_{s,k}(\underline{x})$ is a polynomial of degree s in the variable $\underline{x}$ with real coefficients depending on k. More precisely, $H_{2s,k}(\underline{x})$ only contains even powers of $\underline{x}$ and hence is scalar-valued, while $H_{2s+1,k}(\underline{x})$ only contains odd ones and thus is vector-valued.

3. The Clifford-Fourier transform

In [1] a new multi-dimensional Fourier transform in the framework of Clifford analysis, the so-called Clifford-Fourier transform, is introduced. The idea behind its definition originates from an alternative representation for the standard tensorial multi-dimensional Fourier transform given by

$$\mathcal{F}[f](\underline{\xi}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right)\ f(\underline{x})\ dV(\underline{x}) \quad , \tag{5}$$

where $dV(\underline{x})$ stands for the Lebesgue measure on $\mathbb{R}^m$.
It is indeed so that this classical Fourier transform can be seen as the operator exponential

$$\mathcal{F} = \exp\left(-i\ \frac{\pi}{2}\ \mathcal{H}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-i\frac{\pi}{2}\right)^n \mathcal{H}^n \tag{6}$$

where $\mathcal{H}$ is the scalar-valued differential operator $\mathcal{H} = \frac{1}{2}(-\Delta_m + r^2 - m)$. The equivalence of this operator exponential form with the traditional integral form in (5) may be proved rather easy in the Clifford analysis setting. To this end, we use the orthonormal basis (4) of the space $L_2(\mathbb{R}^m)$. In [10] we have shown that the $L_2(\mathbb{R}^m)$-basis functions $\phi_{s,k,j}$ are simultaneous eigenfunctions of the Fourier transform $\mathcal{F}$ in integral form and of the operator $\mathcal{H}$; more precisely we have at the same time

$$\begin{aligned}\mathcal{F}[\phi_{s,k,j}](\underline{\xi}) &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right) \phi_{s,k,j}(\underline{x})\, dV(\underline{x}) \\ &= \exp\left(-i(s+k)\frac{\pi}{2}\right) \phi_{s,k,j}(\underline{\xi})\end{aligned}$$

and

$$\mathcal{H}[\phi_{s,k,j}(\underline{x})] = (s+k)\ \phi_{s,k,j}(\underline{x}) \quad .$$

It then follows that

$$\begin{aligned}\exp\left(-i\frac{\pi}{2}\mathcal{H}\right)[\phi_{s,k,j}] &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-i\frac{\pi}{2}\right)^n \mathcal{H}^n[\phi_{s,k,j}] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-i\frac{\pi}{2}\right)^n (s+k)^n\ \phi_{s,k,j} \\ &= \exp\left(-i\frac{\pi}{2}(s+k)\right) \phi_{s,k,j} \\ &= \mathcal{F}[\phi_{s,k,j}] \quad ,\end{aligned}$$

which immediately gives rise to the desired equivalence in $L_2(\mathbb{R}^m)$.
Note that, due to the scalar character of the standard Fourier kernel, the Fourier spectrum inherits its Clifford algebra character from the original signal, without any interaction with the Fourier kernel. So in order to genuinely introduce the Clifford analysis character in the Fourier transform, the idea occurred to us to replace the scalar-valued operator $\mathcal{H}$ in the operator exponential (6) by a Clifford algebra-valued one. To that end we aimed at factorizing the operator $\mathcal{H}$, making use of the factorization of the Laplace operator by the Dirac operator. Splitting $\mathcal{H}$ into a sum of Clifford algebra-valued second order operators, leads in a natural way to a *pair* of transforms

$$\mathcal{F}_{\mathcal{H}^\pm} = \exp\left(-i\frac{\pi}{2}\mathcal{H}^\pm\right) \qquad \text{with} \qquad \mathcal{H}^\pm = \mathcal{H} \pm \Gamma \quad ,$$

the geometric average of which is precisely the standard Fourier transform $\mathcal{F}$, i.e. $\mathcal{F}^2 = \mathcal{F}_{\mathcal{H}^+}\ \mathcal{F}_{\mathcal{H}^-}$.

Particular attention is directed towards the two-dimensional case, since then the Clifford-Fourier kernel can be written in a closed form. Indeed, the two-dimensional Clifford-Fourier transform may be expressed as

$$\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp\left(\pm(\underline{\xi} \wedge \underline{x})\right) f(\underline{x})\, dV(\underline{x})$$

with $\exp(\underline{\xi} \wedge \underline{x}) = \sum_{r=0}^{\infty} \frac{(\underline{\xi}\wedge\underline{x})^r}{r!}$. This closed form has enabled us to generalize the well-known results for the standard Fourier transform both in the L_1 and in the L_2-context (see [2]). Note that we have not succeeded yet in obtaining such a closed form in arbitrary dimension.

4. The Fourier-Bessel Transform

In [3] we have introduced another promising multi-dimensional integral transform within the language of Clifford-analysis, the so-called *Fourier-Bessel transform* given by

$$\mathcal{F}_{\mathrm{bes}}[f](\underline{\xi}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} J(\underline{x} \wedge \underline{\xi}) \; f(\underline{x}) \; dV(\underline{x}) \quad .$$

Its integral kernel

$$J(\underline{x}\wedge\underline{\xi}) = 2^{(m-3)/2} \, \Gamma\left(\frac{m-1}{2}\right) \; |\underline{x}\wedge\underline{\xi}|^{(3-m)/2} \left(J_{(m-3)/2}(|\underline{x} \wedge \underline{\xi}|) + \frac{\underline{x} \wedge \underline{\xi}}{|\underline{x} \wedge \underline{\xi}|} \; J_{(m-1)/2}(|\underline{x} \wedge \underline{\xi}|) \right)$$

with J_ν the Bessel function of the first kind, is obtained by leaving out the exponential factor $\exp\left(< \underline{x}, \underline{\xi} >\right)$ from the so-called Bessel-exponential function, introduced by Sommen who recently used it to study Clifford generalizations of the classical Fourier-Borel transform (see [11]).

In the special case of dimension two, it is observed that the Clifford-Fourier and the Fourier-Bessel transform coincide:

$$\mathcal{F}_{bes}[f](\underline{\xi}) = \mathcal{F}_{\mathcal{H}^-}[f](\underline{\xi}) = \mathcal{F}_{\mathcal{H}^+}[f](-\underline{\xi}) \quad .$$

Moreover, the Fourier-Bessel transform satisfies operational formulae which are similar to those of the classical multi-dimensional Fourier transform (5). For example, let us state the differentiation and multiplication rule.

Proposition 1 (differentiation and multiplication rule).
The Fourier-Bessel transform satisfies
(i) the differentiation rule

$$\mathcal{F}_{bes}\left[\partial_{\underline{x}}[f(\underline{x})]\right](\underline{\xi}) = -\underline{\xi} \; \mathcal{F}_{bes}[f(\underline{x})](-\underline{\xi})$$

(ii) the multiplication rule

$$\mathcal{F}_{bes}[\underline{x} f(\underline{x})](\underline{\xi}) = -\partial_{\underline{\xi}}\left[\mathcal{F}_{bes}[f(\underline{x})](-\underline{\xi})\right] \quad .$$

For further use, let us also mention the reflection property.

Proposition 2 (reflection property).
The Fourier-Bessel transform satisfies

$$\mathcal{F}_{bes}[f(-\underline{x})](\underline{\xi}) = \mathcal{F}_{bes}[f(\underline{x})](-\underline{\xi}) \quad .$$

Furthermore, in [3] we have calculated the Fourier-Bessel spectrum of the L_2-basis (4) consisting of generalized Clifford-Hermite functions. We have found that these L_2-basis elements are eigenfunctions of the Fourier-Bessel transform. Indeed, for k even we have obtained that:

$$\mathcal{F}_{bes}\left[H_{2p,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})$$
$$= \frac{(-1)^{k/2}\sqrt{\pi}\ \Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{-k+1}{2}\right)\Gamma\left(\frac{k+m-1}{2}\right)}\ (-1)^p\ H_{2p,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)$$

and

$$\mathcal{F}_{bes}\left[H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})$$
$$= \frac{(-1)^{k/2}\sqrt{\pi}\ \Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{1-k}{2}\right)\Gamma\left(\frac{k+m-1}{2}\right)}\ (-1)^p\ H_{2p+1,k}(\sqrt{2}\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\ P_k(\underline{\xi}) \quad ,$$

while for k odd:

$$\mathcal{F}_{bes}\left[H_{2p,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})$$
$$= \frac{(-1)^{(k-1)/2}\sqrt{\pi}\ \Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k+m}{2}\right)}\ (-1)^p\ H_{2p,k}(\sqrt{2}\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\ P_k(\underline{\xi})$$

and

$$\mathcal{F}_{bes}\left[H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})$$
$$= \frac{(-1)^{(k+1)/2}\sqrt{\pi}\ \Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k+m}{2}\right)}\ (-1)^p\ H_{2p+1,k}(\sqrt{2}\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\ P_k(\underline{\xi}) \quad .$$

The above results will be used in the next section to express the Clifford-Fourier in terms of the Fourier-Bessel transform.

5. The Clifford-Fourier transform in terms of the Fourier-Bessel transform in the even dimensional case

In this section, we will compare the Fourier-Bessel transform with the Clifford-Fourier transform. To this end we will use the following eigenvalue equations (see e.g. [8])

$$\mathcal{F}_{\mathcal{H}^+}\left[H_{2p,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) = (-1)^p\ H_{2p,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)$$

$$\mathcal{F}_{\mathcal{H}^+}\left[H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})$$
$$= (-1)^{p+k}\ (-i)^m\ H_{2p+1,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)$$

$$\mathcal{F}_{\mathcal{H}^-}\left[H_{2p,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) = (-1)^{p+k}\ H_{2p,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)$$

$$\begin{aligned}\mathcal{F}_{\mathcal{H}^-}&\left[H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\\ &= (-1)^{p+1}\ i^m\ H_{2p+1,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) .\end{aligned}$$

The following results only hold in the case of **even dimension** m.

5.1. Basis function $\phi_{2p,k,j}$ with k even

Taking into account the formulae

$$\Gamma(n+z) = z\ (z+1)\ldots(z+n-1)\ \Gamma(z) \quad , \quad \Gamma\left(\frac{1}{2}+z\right)\ \Gamma\left(\frac{1}{2}-z\right) = \frac{\pi}{\cos(\pi z)} \tag{7}$$

with $n \in \mathbb{N}$, we have in the case where m is even that

$$\begin{aligned}&\Gamma\left(\frac{k+m-1}{2}\right)\ \Gamma\left(\frac{-k+1}{2}\right)\\ &\quad = \Gamma\left(\frac{k+1}{2}+\frac{m-2}{2}\right)\ \Gamma\left(\frac{-k+1}{2}\right)\\ &\quad = \left(\frac{k+1}{2}\right)\ \left(\frac{k+3}{2}\right)\ldots\left(\frac{k+m-3}{2}\right)\ \Gamma\left(\frac{1}{2}+\frac{k}{2}\right)\ \Gamma\left(\frac{1}{2}-\frac{k}{2}\right)\\ &\quad = 2^{1-m/2}\ (k+1)(k+3)\ldots(k+m-3)\ \pi\ (-1)^{k/2} .\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\mathcal{F}_{bes}\left[H_{2p,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\\ &\quad = \frac{(-1)^{k/2}\ \sqrt{\pi}\ \Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{-k+1}{2}\right)\ \Gamma\left(\frac{k+m-1}{2}\right)}\ (-1)^p\ H_{2p,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\\ &\quad = \frac{2^{m/2-1}\ \Gamma\left(\frac{m-1}{2}\right)}{\sqrt{\pi}}\ \frac{(-1)^p}{\mathcal{Q}(k)}\ H_{2p,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) ,\end{aligned}$$

where we have introduced the following polynomial of degree $m/2-1$ in k :

$$\mathcal{Q}(k) = (k+m-3)(k+m-5)\ldots(k+3)(k+1) = \frac{(k+m-3)!!}{(k-1)!!} .$$

In view of (2), we have that

$$-\Gamma_{\underline{\xi}}\left[H_{2p,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\right] = k\ H_{2p,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) ,$$

which means also that:

$$\mathcal{Q}(-\Gamma_{\underline{\xi}}) \left[H_{2p,k}(\sqrt{2}\underline{\xi}) \ P_k(\underline{\xi}) \ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\right] = \mathcal{Q}(k) \ H_{2p,k}(\sqrt{2}\underline{\xi}) \ P_k(\underline{\xi}) \ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) \quad .$$

This in its turn, implies that

$$\begin{aligned}
&\mathcal{Q}(-\Gamma_{\underline{\xi}}) \left[\mathcal{F}_{bes} \left[H_{2p,k}(\sqrt{2}\underline{x}) \ P_k(\underline{x}) \ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\right] \\
&\quad = \frac{2^{m/2-1} \ \Gamma\left(\frac{m-1}{2}\right)}{\sqrt{\pi}} \ (-1)^p \ H_{2p,k}(\sqrt{2}\underline{\xi}) \ P_k(\underline{\xi}) \ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) \\
&\quad = \frac{2^{m/2-1} \ \Gamma\left(\frac{m-1}{2}\right)}{\sqrt{\pi}} \mathcal{F}_{\mathcal{H}^\pm} \left[H_{2p,k}(\sqrt{2}\underline{x}) \ P_k(\underline{x}) \ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) \quad .
\end{aligned}$$

We have thus proved that

$$\begin{aligned}
&\mathcal{F}_{\mathcal{H}^\pm} \left[H_{2p,k}(\sqrt{2}\underline{x}) \ P_k(\underline{x}) \ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) \\
&\quad = \frac{\sqrt{\pi}}{2^{m/2-1} \ \Gamma\left(\frac{m-1}{2}\right)} \ \mathcal{Q}(-\Gamma_{\underline{\xi}}) \left[\mathcal{F}_{bes} \left[H_{2p,k}(\sqrt{2}\underline{x}) \ P_k(\underline{x}) \ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\right] \quad .
\end{aligned}$$

Remark 1. *To study the case where m is odd, we use the following formulae satisfied by the Gamma function:*

$$\Gamma(n+1) = n! \qquad \text{and} \qquad \Gamma\left(\frac{1}{2} - n\right) = (-1)^n \ \sqrt{\pi} \ 2^{2n} \ \frac{n!}{(2n)!}$$

with $n \in \mathbb{N}$. We find consecutively

$$\begin{aligned}
&\Gamma\left(\frac{k+m-1}{2}\right) \ \Gamma\left(\frac{-k+1}{2}\right) \\
&\quad = \left(\frac{k+m-3}{2}\right) \ \left(\frac{k+m-5}{2}\right) \ \ldots 2.1 \ (-1)^{k/2} \ \sqrt{\pi} \ 2^k \ \frac{\left(\frac{k}{2}\right)!}{k!} \\
&\quad = \frac{1}{2^{(k+m-3)/2}} \ (k+m-3)!! \ (-1)^{k/2} \ \sqrt{\pi} \ 2^k \ \frac{k(k-2)\ldots 2}{2^{k/2} \ k(k-1)\ldots 1} \\
&\quad = 2^{(3-m)/2} \frac{(k+m-3)!! \ (-1)^{k/2} \ \sqrt{\pi}}{(k-1)!!} \quad .
\end{aligned}$$

Hence, in this case the expression

$$\frac{(-1)^{k/2} \ \sqrt{\pi} \ \Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{-k+1}{2}\right) \ \Gamma\left(\frac{k+m-1}{2}\right)} = \frac{\Gamma\left(\frac{m-1}{2}\right) \ (k-1)!!}{(k+m-3)!!} \ 2^{(m-3)/2}$$

yields a rational function in k. It follows that the previous method cannot be applied in case where m is odd.

5.2. Basis function $\phi_{2p,k,j}$ with k odd

Again using the formulae (7) we find that for m even and k odd:

$$\begin{aligned}
&\Gamma\left(\frac{k+m}{2}\right)\ \Gamma\left(-\frac{k}{2}\right)\\
&\quad = \ \Gamma\left(\frac{m-2}{2}+\frac{k+2}{2}\right)\ \Gamma\left(\frac{1}{2}-\frac{k+1}{2}\right)\\
&\quad = \ \left(\frac{k+2}{2}\right)\ \left(\frac{k+4}{2}\right)\ldots\left(\frac{k+m-2}{2}\right)\ \Gamma\left(\frac{1}{2}+\frac{k+1}{2}\right)\ \Gamma\left(\frac{1}{2}-\frac{k+1}{2}\right)\\
&\quad = \ \frac{1}{2^{(m-2)/2}}\ (k+2)(k+4)\ldots(k+m-2)\ \frac{\pi}{\cos\left(\pi\left(\frac{k+1}{2}\right)\right)}\\
&\quad = \ \frac{1}{2^{(m-2)/2}}\ \frac{(k+m-2)!!}{k!!}\ \pi\ (-1)^{(k+1)/2}\quad ,
\end{aligned}$$

which yields

$$\begin{aligned}
&\mathcal{F}_{bes}\left[H_{2p,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\\
&\quad = \ \frac{(-1)^{(k-1)/2}\ \sqrt{\pi}\ \Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(-\frac{k}{2}\right)\ \Gamma\left(\frac{k+m}{2}\right)}\ (-1)^p\ H_{2p,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\\
&\quad = \ \frac{2^{m/2-1}\ \Gamma\left(\frac{m-1}{2}\right)}{\sqrt{\pi}}\ \frac{(-1)^{p+1}\ k!!}{(k+m-2)!!}\ H_{2p,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\\
&\quad = \ \frac{2^{m/2-1}\ \Gamma\left(\frac{m-1}{2}\right)}{\sqrt{\pi}}\ \frac{(-1)^{p+1}}{\mathcal{Q}(k+1)}\ H_{2p,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\quad .
\end{aligned}$$

This also implies that

$$\begin{aligned}
&\mathcal{Q}(-\Gamma_{\underline{\xi}}+I)\left[\mathcal{F}_{bes}\left[H_{2p,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\right]\\
&\quad = \ \frac{2^{m/2-1}\ \Gamma\left(\frac{m-1}{2}\right)}{\sqrt{\pi}}\ (-1)^{p+1}\ H_{2p,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\\
&\quad = \ \pm\ \frac{2^{m/2-1}\ \Gamma\left(\frac{m-1}{2}\right)}{\sqrt{\pi}}\mathcal{F}_{\mathcal{H}^\mp}\left[H_{2p,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\quad ,
\end{aligned}$$

where I denotes the identity operator. Hence, in the case where k is odd, we have shown that

$$\begin{aligned}
&\mp\ \mathcal{F}_{\mathcal{H}^\pm}\left[H_{2p,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\\
&\quad = \frac{\sqrt{\pi}}{2^{m/2-1}\Gamma\left(\frac{m-1}{2}\right)}\ \mathcal{Q}(-\Gamma_{\underline{\xi}}+I)\left[\mathcal{F}_{bes}\left[H_{2p,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\right]\quad .
\end{aligned}$$

5.3. Basis function $\phi_{2p+1,k,j}$ with k even

Similarly as in subsection 5.1, we have that

$$\begin{aligned}
&\mathcal{F}_{bes}\left[H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) \\
&\qquad = \frac{(-1)^{k/2}\ \sqrt{\pi}\ \Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{1-k}{2}\right)\ \Gamma\left(\frac{k+m-1}{2}\right)}\ (-1)^p\ H_{2p+1,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) \\
&\qquad = \frac{2^{m/2-1}\ \Gamma\left(\frac{m-1}{2}\right)}{\sqrt{\pi}}\ \frac{(-1)^p}{\mathcal{Q}(k)}\ H_{2p+1,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) \quad .
\end{aligned}$$

In view of (3), we find that

$$\begin{aligned}
&(-\Gamma_{\underline{\xi}} + I)\left[H_{2p+1,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\right] \\
&\qquad\qquad = (-k-m+2)\ H_{2p+1,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) \quad ,
\end{aligned}$$

which gives

$$\begin{aligned}
&(-1)^{m/2-1}\ \mathcal{Q}(-\Gamma_{\underline{\xi}} + I)\left[\mathcal{F}_{bes}\left[H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\right] \\
&= \frac{(-1)^{m/2-1}\ 2^{m/2-1}\ \Gamma\left(\frac{m-1}{2}\right)\ (-1)^p}{\sqrt{\pi}\ \mathcal{Q}(k)}\ \mathcal{Q}(-k-m+2)\ H_{2p+1,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) \\
&= \frac{2^{m/2-1}\ \Gamma\left(\frac{m-1}{2}\right)}{\sqrt{\pi}}\ (-1)^p\ H_{2p+1,k}(\sqrt{2}\underline{\xi})\ P_k(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) \quad ,
\end{aligned}$$

since

$$\begin{aligned}
\mathcal{Q}(-k-m+2) &= (-k-1)(-k-3)\ldots(-k-m+5)(-k-m+3) \\
&= (-1)^{m/2-1}\ (k+m-3)(k+m-5)\ldots(k+3)(k+1) \\
&= (-1)^{m/2-1}\ \mathcal{Q}(k) \quad .
\end{aligned}$$

Hence we arrive at

$$\begin{aligned}
&(-1)^{m/2-1}\ \mathcal{Q}(-\Gamma_{\underline{\xi}} + I)\left[\mathcal{F}_{bes}\left[H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\right] \\
&\qquad = \frac{2^{m/2-1}\ \Gamma\left(\frac{m-1}{2}\right)}{\sqrt{\pi}}\ (\pm 1)\ (\pm i)^m\ \mathcal{F}_{\mathcal{H}^\pm}\left[H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) \quad .
\end{aligned}$$

As $(\pm i)^m = (-1)^{m/2}$, this can be rewritten as

$$\begin{aligned}
&\mp\ \mathcal{F}_{\mathcal{H}^\pm}\left[H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) \\
&\qquad = \frac{\sqrt{\pi}}{2^{m/2-1}\ \Gamma\left(\frac{m-1}{2}\right)}\ \mathcal{Q}(-\Gamma_{\underline{\xi}} + I)\left[\mathcal{F}_{bes}\left[H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\right] \quad .
\end{aligned}$$

5.4. Basis function $\phi_{2p+1,k,j}$ with k odd

In this case we find for m even

$$\mathcal{F}_{\mathcal{H}^\pm}\left[H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})$$
$$= \frac{\sqrt{\pi}}{2^{m/2-1}\ \Gamma\left(\frac{m-1}{2}\right)}\ \mathcal{Q}(-\Gamma_{\underline{\xi}})\left[\mathcal{F}_{bes}\left[H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\right] \quad .$$

5.5. General expression

Summarizing, we can state that in case of dimension m even:

A) <u>k, s: same parity</u>

$$\mathcal{F}_{\mathcal{H}^\pm}\left[H_{s,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})$$
$$= c_m\ \mathcal{Q}(-\Gamma_{\underline{\xi}})\left[\mathcal{F}_{bes}\left[H_{s,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\right]$$

B) <u>k, s: different parity</u>

$$\mp\ \mathcal{F}_{\mathcal{H}^\pm}\left[H_{s,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})$$
$$= c_m\ \mathcal{Q}(-\Gamma_{\underline{\xi}}+I)\left[\mathcal{F}_{bes}\left[H_{s,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})\right]$$

with

$$\mathcal{Q}(k) = (k+1)(k+3)\ldots(k+m-5)(k+m-3) \quad \text{and} \quad c_m = \frac{\sqrt{\pi}}{2^{m/2-1}\ \Gamma\left(\frac{m-1}{2}\right)} \quad .$$

Note that the above result only depends on the even or odd character of the basis function.

As each function $f \in L_2(\mathbb{R}^m)$ can be decomposed into its even and odd part:

$$\begin{aligned} f(\underline{x}) &= f^e(\underline{x}) + f^o(\underline{x}) \\ &= \frac{1}{2}(I+S)[f](\underline{x}) + \frac{1}{2}(I-S)[f](\underline{x}) \end{aligned}$$

with S the "antipodal" map given by

$$S[f](\underline{x}) = f(-\underline{x}) \quad ,$$

we have that:

$$\begin{aligned} \mathcal{F}_{\mathcal{H}^+}\ \frac{1}{2}(I+S) &= c_m\ \mathcal{Q}(-\Gamma_{\underline{\xi}})\ \mathcal{F}_{bes}\ \frac{1}{2}(I+S) \\ \mathcal{F}_{\mathcal{H}^+}\ \frac{1}{2}(I-S) &= -c_m\ \mathcal{Q}(-\Gamma_{\underline{\xi}}+I)\ \mathcal{F}_{bes}\ \frac{1}{2}(I-S) \quad . \end{aligned}$$

Moreover, taking into account the reflection property of the Fourier-Bessel transform (see Proposition 2), it is easily seen that the Fourier-Bessel transform and the antipodal map S commute, i.e. $\mathcal{F}_{bes}\ S = S\ \mathcal{F}_{bes}$.
Hence, we can also write

$$\begin{aligned}\mathcal{F}_{\mathcal{H}^+}\ \frac{1}{2}(I+S) &= c_m\ \mathcal{Q}(-\Gamma_{\underline{\xi}})\ \frac{1}{2}(I+S)\ \mathcal{F}_{bes} \\ \mathcal{F}_{\mathcal{H}^+}\ \frac{1}{2}(I-S) &= -c_m\ \mathcal{Q}(-\Gamma_{\underline{\xi}}+I)\ \frac{1}{2}(I-S)\ \mathcal{F}_{bes} \quad .\end{aligned}$$

Adding the above equations finally yields

$$\mathcal{F}_{\mathcal{H}^+} = c_m \left\{ \mathcal{Q}(-\Gamma_{\underline{\xi}}) \left(\frac{I+S}{2}\right) - \mathcal{Q}(-\Gamma_{\underline{\xi}}+I) \left(\frac{I-S}{2}\right) \right\}\ \mathcal{F}_{bes} \quad . \tag{8}$$

Similarly, we find

$$\mathcal{F}_{\mathcal{H}^-} = c_m \left\{ \mathcal{Q}(-\Gamma_{\underline{\xi}}) \left(\frac{I+S}{2}\right) + \mathcal{Q}(-\Gamma_{\underline{\xi}}+I) \left(\frac{I-S}{2}\right) \right\}\ \mathcal{F}_{bes} \quad . \tag{9}$$

6. Closed form for the Clifford-Fourier integral kernel in the even dimensional case

By means of equations (8) and (9), we immediately obtain the following result.

Theorem 1.
The integral kernel of the Clifford-Fourier transform $\mathcal{F}_{\mathcal{H}^\pm}$ in case of even dimension is given by

$$c_m \left\{ \mathcal{Q}(-\Gamma_{\underline{\xi}}) \left(\frac{I+S}{2}\right) [J(\underline{x}\wedge\underline{\xi})] \mp \mathcal{Q}(-\Gamma_{\underline{\xi}}+I) \left(\frac{I-S}{2}\right) [J(\underline{x}\wedge\underline{\xi})] \right\} \tag{10}$$

with

$$\mathcal{Q}(k) = (k+1)(k+3)\ldots(k+m-5)(k+m-3) \quad \text{and} \quad c_m = \frac{\sqrt{\pi}}{2^{m/2-1}\ \Gamma\left(\frac{m-1}{2}\right)} \quad .$$

The even and odd part of the Fourier-Bessel kernel are respectively scalar- and bivector-valued:

$$\begin{aligned}\left(\frac{I+S}{2}\right)[J(\underline{x}\wedge\underline{\xi})] &= 2^{(m-3)/2}\ \Gamma\left(\frac{m-1}{2}\right)\ |\underline{x}\wedge\underline{\xi}|^{(3-m)/2}\ J_{(m-3)/2}(|\underline{x}\wedge\underline{\xi}|) \\ \left(\frac{I-S}{2}\right)[J(\underline{x}\wedge\underline{\xi})] &= 2^{(m-3)/2}\ \Gamma\left(\frac{m-1}{2}\right)\ |\underline{x}\wedge\underline{\xi}|^{(1-m)/2}\ J_{(m-1)/2}(|\underline{x}\wedge\underline{\xi}|)\ (\underline{x}\wedge\underline{\xi}) \quad .\end{aligned}$$

6.1. Case $m = 2$

In the case where $m = 2$, we have that $\mathcal{Q}(k) = c_2 = 1$.
Taking into account that (see e.g. [12])

$$J_{1/2}(z) = \left(\frac{\pi}{2}z\right)^{-1/2} \sin(z) \qquad \text{and} \qquad J_{-1/2}(z) = \left(\frac{\pi}{2}z\right)^{-1/2} \cos(z) \quad ,$$

expression (10) indeed yields the two-dimensional Clifford-Fourier kernel:

$$\begin{aligned} &\sqrt{\frac{\pi}{2}}\ |\underline{x} \wedge \underline{\xi}|^{1/2}\ J_{-1/2}(|\underline{x} \wedge \underline{\xi}|) \mp \sqrt{\frac{\pi}{2}} \frac{(\underline{x} \wedge \underline{\xi})}{|\underline{x} \wedge \underline{\xi}|^{1/2}}\ J_{1/2}(|\underline{x} \wedge \underline{\xi}|) \\ &\qquad = \cos(|\underline{x} \wedge \underline{\xi}|) \mp \frac{\underline{x} \wedge \underline{\xi}}{|\underline{x} \wedge \underline{\xi}|}\ \sin(|\underline{x} \wedge \underline{\xi}|) = \exp\left(\mp(\underline{x} \wedge \underline{\xi})\right) \quad . \end{aligned}$$

6.2. Case $m = 4$

Let us now calculate the Clifford-Fourier kernels in case $m = 4$.
First, we have that $c_4 = 1$ and $\mathcal{Q}(k) = k + 1$.
Next, we obtain

$$\Gamma_{\underline{\xi}}[\underline{\xi} \wedge \underline{x}] = \Gamma_{\underline{\xi}}[\underline{\xi}\underline{x} + < \underline{\xi}, \underline{x} >] \ = \ 3\ \underline{\xi}\underline{x} + \underline{x} \wedge \underline{\xi} = -3 < \underline{\xi}, \underline{x} > +2\ (\underline{\xi} \wedge \underline{x}) \quad , \tag{11}$$

since

$$\Gamma_{\underline{\xi}}[\underline{\xi}] = (m-1)\underline{\xi} \qquad \text{and} \qquad \Gamma_{\underline{\xi}}[< \underline{\xi}, \underline{x} >] = \underline{x} \wedge \underline{\xi} \quad .$$

Furthermore, in view of (1) we also find that

$$\begin{aligned} \Gamma_{\underline{\xi}}[|\underline{x} \wedge \underline{\xi}|^2] &= \Gamma_{\underline{\xi}}[|\underline{x}|^2|\underline{\xi}|^2 - (< \underline{x}, \underline{\xi} >)^2] \ = \ -2 < \underline{x}, \underline{\xi} > \ \Gamma_{\underline{\xi}}[< \underline{x}, \underline{\xi} >] \\ &= -2\ < \underline{x}, \underline{\xi} >\ (\underline{x} \wedge \underline{\xi}) \quad . \end{aligned}$$

Combining the above result with

$$\Gamma_{\underline{\xi}}[|\underline{x} \wedge \underline{\xi}|^2] = 2|\underline{x} \wedge \underline{\xi}|\ \Gamma_{\underline{\xi}}[|\underline{x} \wedge \underline{\xi}|] \quad ,$$

yields

$$\Gamma_{\underline{\xi}}[|\underline{x} \wedge \underline{\xi}|] = \frac{< \underline{x}, \underline{\xi} >\ (\underline{\xi} \wedge \underline{x})}{|\underline{x} \wedge \underline{\xi}|} \quad . \tag{12}$$

Taking into account formula (12) and the following formulae (see e.g. [12])

$$J'_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z}\ J_\nu(z) \qquad \text{and} \qquad J_{\nu-1}(z) - 2\ \frac{\nu}{z}\ J_\nu(z) = -J_{\nu+1}(z) \quad ,$$

we arrive at

$$\begin{aligned} &\mathcal{Q}(-\Gamma_{\underline{\xi}}) \left(\frac{I+S}{2}\right) [J(\underline{x} \wedge \underline{\xi})] \\ &\qquad = (-\Gamma_{\underline{\xi}} + I) \left[\sqrt{\frac{\pi}{2}}\ |\underline{x} \wedge \underline{\xi}|^{-1/2}\ J_{1/2}(|\underline{x} \wedge \underline{\xi}|)\right] \\ &\qquad = \sqrt{\frac{\pi}{2}}\ |\underline{x} \wedge \underline{\xi}|^{-1/2} \left(J_{1/2}(|\underline{x} \wedge \underline{\xi}|) + \frac{(\underline{\xi} \wedge \underline{x})}{|\underline{x} \wedge \underline{\xi}|}\ J_{3/2}(|\underline{x} \wedge \underline{\xi}|)\ < \underline{x}, \underline{\xi} >\right) \quad . \end{aligned}$$

Similarly, a straightforward computation using also formula (11), yields

$$\begin{aligned}\mathcal{Q}(-\Gamma_{\underline{\xi}}+I)\left(\frac{I-S}{2}\right)[J(\underline{x}\wedge\underline{\xi})] &= (-\Gamma_{\underline{\xi}}+2I)\left[\sqrt{\frac{\pi}{2}}\ |\underline{x}\wedge\underline{\xi}|^{-3/2}\ J_{3/2}(|\underline{x}\wedge\underline{\xi}|)\ (\underline{x}\wedge\underline{\xi})\right] \\ &= -\sqrt{\frac{\pi}{2}}\ |\underline{x}\wedge\underline{\xi}|^{-1/2}\ J_{1/2}(|\underline{x}\wedge\underline{\xi}|)\ <\underline{x},\underline{\xi}> \quad .\end{aligned}$$

Hence for $m = 4$ we have derived the following closed form of the kernel of $\mathcal{F}_{\mathcal{H}^\pm}$:

$$\sqrt{\frac{\pi}{2}}\ |\underline{x}\wedge\underline{\xi}|^{-1/2}\left((1\pm<\underline{x},\underline{\xi}>)\ J_{1/2}(|\underline{x}\wedge\underline{\xi}|)+\frac{(\underline{\xi}\wedge\underline{x})}{|\underline{x}\wedge\underline{\xi}|}\ J_{3/2}(|\underline{x}\wedge\underline{\xi}|)\ <\underline{x},\underline{\xi}>\right) \quad .$$

6.3. Case $m = 6$

In case of dimension $m = 6$, we obtain after a similar lengthy calculation the following kernels of $\mathcal{F}_{\mathcal{H}^\pm}$:

$$\begin{aligned}&\sqrt{\frac{\pi}{2}}\left\{|\underline{x}\wedge\underline{\xi}|^{-1/2}\left(J_{1/2}(|\underline{x}\wedge\underline{\xi}|)+\frac{<\underline{x},\underline{\xi}>^2}{|\underline{x}\wedge\underline{\xi}|}\ J_{3/2}(|\underline{x}\wedge\underline{\xi}|)\right)\ \pm\ |\underline{x}\wedge\underline{\xi}|^{-3/2}\right. \\ &\left.\left(2<\underline{x},\underline{\xi}>\ J_{3/2}(|\underline{x}\wedge\underline{\xi}|)+(\underline{\xi}\wedge\underline{x})\ J_{3/2}(|\underline{x}\wedge\underline{\xi}|)+\frac{<\underline{x},\underline{\xi}>^2}{|\underline{x}\wedge\underline{\xi}|}\ (\underline{\xi}\wedge\underline{x})\ J_{5/2}(|\underline{x}\wedge\underline{\xi}|)\right)\right\} \quad .\end{aligned}$$

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